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Processing process: The Gilbreath conjecture

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ABSTRACT

There is one very important aspect of history that is often left out – the process. The development of the Gilbreath conjecture is described as an example of this issue. This includes a theorem that delineates the possible series of integers satisfying the conditions of the conjecture. Those who do not learn from history are destined to have less options. Processing a process can suggest relevant questions, it will increase our options – which is an important part of any exploration.

1. Introduction

There is one very important aspect of history that is often left out – the process. This is even true of the history of mathematics. I will give an example. A number of years ago (around 1958) I developed a number theory conjecture concerning the primes. This is known in number theory as the Gilbreath conjecture. It is easy to state but even though the great number theorist Erdos believed it was true, he also believed it would take about 200 years to prove.

Who am I to doubt Erdos? However, it seems no one has seriously considered why and how I came up with this conjecture. So, I will now describe the process, and some observations suggested by this process – to hopefully show why processing process is important.

2. Theory

First, a statement of the conjecture as I originally presented it:

If the first n primes (starting with 2) are placed in a row, and more rows are formed by taking the absolute difference between each consecutive pair of the previous row, then every row after the row of primes will start with a one.

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For example:

Columns:	1	2	3	4	5	6
Row 1:	2	3	5	7	11	13
Row 2:	1	2	2	4	2	
Row 3:	1	0	2	2		
Row 4:	1	2	0			
Row 5:	1	2				
Row 6:	1					

I originally presented the conjecture this way to make it easier to grasp; however, this is not how I first dealt with the conjecture. My first representation to myself was as follows:

If the first n primes (starting with 2) are placed in a row, and more rows are formed, by subtracting the absolute value of the first of each consecutive pair of the previous row from the absolute value of the second of the pair, then every row after the row of primes will start with a $+$ or $-$ one.

For example:

Columns:	1	2	3	4	5	6
Row 1:	2	3	5	7	11	13
Row 2:		1	2	2	4	2
Row 3:			1	0	2	−2
Row 4:				−1	2	0
Row 5:					1	−2
Row 6:						1

While these two representations are mathematically equivalent, I prefer the one that states every row after the row of primes starts with a $+$ or $-$ one. To understand why I have this preference let's consider the history of why and how I developed the conjecture.

My aim was to find a way to generate the primes, so I looked for a pattern in the differences between primes. I noticed if I made more rows of differences the resulting numbers were smaller. Then I noticed if I used the difference of the absolute values of the numbers, the resulting number would sometimes be even smaller. Further, if I kept the signs I could always work backwards; I had not lost any information. Working backwards was important; remember I wanted a way to generate the primes. I also noticed the last subtraction below any prime seemed to result in a $+$ or $-$ one.

If every row after the row of primes starts with a $+$ or $-$ one, and I could find a way to generate the complete $+$ or $-$ pattern, then I could generate the primes. Unfortunately I could not find a way to generate the $+$ or $-$ pattern; it may very well be that forming the primes and then all the other rows is the simplest way to generate the $+$ or $-$ pattern.

So I was not successful, but all was not lost – there still was a necessary condition for an integer to be a prime: every row of differences of absolute values must start with a $+$ or $-$ one. Of course this was only a conjecture. To illustrate the value of exploring any process being used in research, I will describe how I have approached this problem, and with what results. The reasons for the conclusions will not be formal proofs, but more like outlines of proofs. The important theorem is Theorem 3, which delineates the contents of row 1, and the first time a non-trivial property of the primes is used is in Theorem 5.

Let's start by describing the general algorithm I use, then give reasons for certain conditions to be applied to the algorithm, and present some resulting theorems.

Algorithm.

$$
\begin{array}{rllllllll}
\text{Columns:} & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\text{Row 1:} & \mathrm{I}_{11} & \mathrm{I}_{12} & \mathrm{I}_{13} & \mathrm{I}_{14} & \mathrm{I}_{15} & \mathrm{I}_{16} & \\
\text{Row 2:} & & \pm\mathrm{D}_{22} & \pm\mathrm{D}_{23} & \pm\mathrm{D}_{24} & \pm\mathrm{D}_{25} & \pm\mathrm{D}_{26} & \\
\text{Row 3:} & & & \pm\mathrm{D}_{33} & \pm\mathrm{D}_{34} & \pm\mathrm{D}_{35} & \pm\mathrm{D}_{36} & \\
\text{Row 4:} & & & & \pm\mathrm{D}_{44} & \pm\mathrm{D}_{45} & \pm\mathrm{D}_{46} & \\
\text{Row 5:} & & & & & \pm\mathrm{D}_{55} & \pm\mathrm{D}_{56} & \\
\text{Row 6:} & & & & & & \pm\mathrm{D}_{66} & \\
\vdots & & & & & & &
\end{array}
$$

$$
\begin{aligned}
\text{where:}\quad & \mathrm{D}_{\mathrm{RC}} \geqslant 0 \\
& \pm\,\mathrm{D}_{2\mathrm{C}} = \mathrm{I}_{1\mathrm{C}} - \mathrm{I}_{1(\mathrm{C}-1)} \\
& \pm\,\mathrm{D}_{(>2)(\geqslant\mathrm{R})} = \mathrm{D}_{(\mathrm{R}-1)\mathrm{C}} - \mathrm{D}_{(\mathrm{R}-1)(\mathrm{C}-1)}
\end{aligned}
$$

The first row is a series of integers, and the following rows are differences obtained for each consecutive pair of the previous row by subtracting the absolute value of the first of the pair from the absolute value of the second of the pair. The results are represented by $+$ or $-$ sign and a positive integer D. (The subscripts represent the row and column.)

3. Results

Theorem 1. I_{11} *and* D_{RR} *and* $+$ *pattern* $\leftrightarrow$ *row* 1.

This theorem states that given the absolute value of the first integer of every row, and given the pattern of $+$ or $-$ then the contents of the first row is determined. Conversely, given the contents of the first row then the contents of all the other rows is determined. This equivalence is determined by the definition of the algorithm. This theorem motivates the three conditions to be applied to the algorithm.

Condition 1: $\mathrm{I}_{11} = 2$
Condition 2: $\mathrm{D}_{\mathrm{RR}} = 1$
Condition 3: $\pm\mathrm{D}_{2\mathrm{C}} > 0$

Condition 1 gives the first integer of row 1 the value 2. If this integer was not 2 but some value K, then $2 - \mathrm{K}$ could be added to every integer in row 1 without any affect on any of the other rows so there is no loss of generality by making the first integer of row $1 = 2$. Besides, 2 is the first prime.

Condition 2 gives the first integer of every row after the first row the absolute value 1. Since we want to understand what series of numbers are generated by the algorithm, it seems making the first integer of each row after the first have the same absolute value of 1 increases the likelihood of obtaining an interesting series of integers in the first row. Besides, ± 1 is what the primes seem to produce.

Condition 3 gives all the integers of the second row a positive value. Since the first integer has been given the value 2, this is equivalent to making all the integers of row 1 be positive and increasing. Besides, the primes are positive and increasing.

This leads us to Theorem 2.

Theorem 2. $I_{11} = 2$ *and* $D_{RR} = 1$ *and* $\pm D_{2C} > 0 \to$

$$\begin{array}{llllll} \textit{Columns}: & 1 & 2 & 3 & 4 & \ldots \\ \textit{Row } 1: & 2 & 3 & 5 & \textit{odd}, > 0, \textit{increasing} & \ldots \\ \textit{Row } 2: & & 1 & 2 & \textit{even}, > 0\text{: } I_{1C} - I_{1(C-1)} & \ldots \\ \textit{Row } 3: & & & 1 & \textit{even}\text{: } D_{2C} - D_{2(C-1)} & \ldots \end{array}$$

$$\begin{array}{llll} \textit{Columns}: & R & R+1 & \ldots \\ \textit{Row } R: & \pm D_{RR} = \pm 1 & \textit{even}\text{: } D_{(R-1)C} - D_{(R-1)(C-1)} & \ldots \end{array}$$

This theorem states that given the three conditions, the rows will have the following properties:

(A) Row 1 starts with the values 2, 3, 5 and row 2 starts with the values 1, 2 and row 3 starts with the value 1.
(B) After row 1 the first integer of every row is + or −1.
(C) After row 1 and after the first integer of each row all the rest of the row will be even.
(D) After column 1 all integers of row 1 are odd.
(E) All integers of row 2 are > 0.
(F) After column 1 all integers of row 1 are > 0 and increasing.

Now for the important theorem, one that delineates the contents of row 1.

Theorem 3. $I_{11} = 2$ *and* $D_{RR} = 1$ *and* $\pm D_{2C} > 0 \to$

$$I_{1C} < I_{1(C+1)} \leqslant I_{1C} + \sum D_{RC} + 1$$

where $\sum D_{RC}$ *means the sum of the absolute values of all the integers of column* C *after* I_{1C}.

This theorem states that given the three conditions, for an integer in row 1 and column C, the next integer in row 1 can be the integer in column C plus any even number up to and including (1 plus the sum of the Ds in column C).

Absolute difference function. Form a series by placing $I_{1(C+1)}$ before the negative of all the integers of column C:

$$(I_{1(C+1)},\ -I_{1C},\ -D_{2C},\ -D_{3C},\ \ldots,\ -D_{(C-1)C},\ -1)$$

Starting at the left, add the members of the series together until the series has ended, or the result is < 0. If the result is < 0 then change its sign and continue adding.

The result is the absolute value for the last integer in column $C + 1$.

Let S represent the increase in the result due to sign changing. Since taking absolute values is the same as changing from − to + the sign in front of each of the Ds which would have changed the sign of the Absolute difference function from + to − at that point, therefore $S = 2\sum D$ for all sign-changing Ds.

Case 1. If the result is 1 then: $I_{1(C+1)} - I_{1C} - \sum D_{RC} + S = 1$, therefore $I_{1(C+1)} \leqslant I_{1C} + \sum D_{RC} + 1$. Since the result is 1 the absolute value of the last integer of column $C + 1$ will be 1. Therefore any integer satisfying Case 1 will satisfy the conditions of Theorem 3.

Case 2. If the result is > 1 then: $I_{1(C+1)} - I_{1C} - \sum D_{RC} > 1$, therefore $I_{1(C+1)} > I_{1C} + \sum D_{RC} + 1$. Since the result is > 1 the absolute value of the last integer of column $C + 1$ will be > 1. Therefore any integer satisfying Case 2 will not satisfy the conditions of Theorem 3.

Case 3. If the result is < 1 then the result would be 0 and the next to the last result would be 1, but since the next to the last result must be an even number, therefore this case cannot happen.

It is clear the row 1 series can be many series of integers that are not all the primes. In fact Theorem 3 shows that every series of integers which could be row 1 can be formed by choosing for each column $C + 1$ ANY of the next $(\sum D_{RC} + 1)/2$ odd integers larger than the integer in column C.

Theorem 4. $I_{11} = 2$ *and* $D_{RR} = 1$ *and* $\pm D_{2C} > 0 \rightarrow$ *a series can be formed which contains all the primes.*

Since I_{1C} and the allowed next integers are consecutive odd integers, therefore, a series can always be formed containing ALL the primes by selecting the next prime in the series of integers that can immediately follow the current prime, unless there is no prime in the allowed series, then continue selecting the maximum next allowed integer until the next prime occurs in an allowed series.

Theorem 5. $I_{11} = 2$ *and* $D_{RR} = 1$ *and* $\pm D_{2C} > 0 \rightarrow$ *a series can be formed containing all the primes and no consecutive non-primes.*

After column C, let $I_{1(C+1)}$ be the maximum next allowed integer, and let $I_{1(C+2)}$ be the maximum next allowed integer after $I_{1(C+1)}$. Let E_{1X} be I_{1X} and $E_{(>1)X}$ be $D_{(>1)X}$.

For $C > 1$

Show $I_{1C} < D_{2(C+2)}$

$D_{2(C+2)} = \sum D_{(>1)(C+1)} + 1$

The approach to deriving row 1 from the array of absolute differences led to working backwards with the following series of inequalities.

Show (for $R = C$ to 1) $E_{RC} < \sum D_{(>R)(C+1)} + 1$

Let $R = C + 1 - X$ (for $X = 1$ to C)

Show (for $X = 1$) $E_{RC} < \sum D_{(>R)(C+1)} + 1$

$E_{CC} = 1$

$\sum D_{(>C)(C+1)} + 1 = 2$

(for $X = 1$) $E_{RC} < \sum D_{(>R)(C+1)} + 1$

Assume (for $X \leqslant N$) $E_{RC} < \sum D_{(>R)(C+1)} + 1$

Show (for $X = N + 1$) $E_{(R-1)C} < \sum D_{(>R-1)(C+1)} + 1$

For any column $\operatorname{Max} E_{RC} = 2^{(X-1)}$

Let $E_{(R-1)C} = \operatorname{Max} E_{(R-1)C}$

(for K from R to C) $D_{KC} = \operatorname{Max} D_{KC}$

(for K from C to R) $D_{K(C+1)} = \operatorname{Max} D_{K(C+1)} = \operatorname{Max} E_{(K-1)C}$

$\operatorname{Max} E_{(R-1)C} = D_{R(C+1)}$

$E_{(R-1)C} \leqslant D_{R(C+1)}$

$$(\text{for } X = N + 1)\ E_{(R-1)C} < \sum D_{(>R-1)(C+1)} + 1$$
$$(\text{for } R = C \text{ to } 1)\ E_{RC} < \sum D_{(>R)(C+1)} + 1$$
$$I_{1C} < \sum D_{(>1)(C+1)} + 1$$
$$I_{1C} < D_{2(C+2)}$$
$$I_{1C} < D_{2(C+1)} + D_{2(C+2)}$$
$$I_{1(C+2)} = I_{1C} + D_{2(C+1)} + D_{2(C+2)}$$
$$I_{1(C+2)} > 2I_{1C}$$

And now for the first time to consider a non-trivial property of the primes. Since there is at least one prime between any integer and twice that integer, therefore there is a prime between I_{1C} and $I_{1(C+2)}$. If there is no prime between I_{1C} and $I_{1(C+1)}$ then there is a prime between $I_{1(C+1)}$ and $I_{1(C+2)}$. Therefore, a series can be formed containing all the primes with no consecutive non-primes by selecting the next prime in the series of integers that can immediately follow the current prime, unless there is no prime in the allowed series, then select the maximum next allowed integer. There will always be a prime in the next series of allowed integers after the selected maximum next integer, therefore there will never be any consecutive non-primes.

4. Conclusions

Since it is not always true that $I_{1(C+1)} \geqslant 2I_{1C}$ the property of the primes just considered only takes us this far. Now you have 200 years to get rid of the non-consecutive non-primes – since my conjecture maintains: If the series in row 1 has been primes up to some point then the series of allowed integers for the next integer in row 1 always includes the next prime. To paraphrase a well-known statement: Those that do not learn from history are destined to have less options. A question that has not been asked cannot be answered – and since processing a process can suggest relevant questions, it will increase our options – which is an important part of any exploration.